Distance labellings of Cayley graphs of semigroups

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Abstract

This paper establishes connections between the structure of a semigroup and the minimum spans of distance labellings of its Cayley graphs. We show that certain general restrictions on the minimum spans are equivalent to the semigroup being combinatorial, and that other restrictions are equivalent to the semigroup being a right zero band. We obtain a description of the structure of all semigroups S and their subsets C such that $\mathrm{Cay}(S, C)$ is a disjoint union of complete graphs, and show that this description is also equivalent to several restrictions on the minimum span of $\mathrm{Cay}(S, C)$. We then describe all graphs with minimum spans satisfying the same restrictions, and give examples to show that a fairly straightforward upper bound for the minimum spans of the underlying undirected graphs of Cayley graphs turns out to be sharp even for the class of combinatorial semigroups.

Keywords: distance labellings, combinatorial semigroups, bands, Cayley graphs, unions of complete graphs

1 Introduction

This paper investigates distance labellings of Cayley graphs of semigroups. Throughout the paper the term *graph* means a finite directed graph without multiple edges, but possibly with loops. A graph $\Gamma = (V, E)$ is said to be *undirected* if and only if, for every $(u, v) \in E$, the edge (v, u) belongs to E. All our theorems will involve undirected Cayley graphs only. Let $\mathbb{N}$ be the set of all positive integers, and let $k_1, k_2, \ldots, k_\ell \in \mathbb{N}$ for some $\ell \geq 1$. A *distance labelling*, or an $L(k_1, k_2, \ldots, k_\ell)$-*labelling* of an undirected graph $\Gamma = (V, E)$ is a mapping $f : V \to \mathbb{N}$ such that $|f(u) - f(v)| \geq k_t$ for $t = 1, 2, \ldots, \ell$ and any $u, v \in V$ with $d(u, v) = t$, where $d(u, v)$ is the distance between u and v in Γ. The integer $f(v)$ is called the *label* assigned to v under f, and the difference between the largest and the smallest labels is called the *span* of f. The minimum span over the set of all $L(k_1, k_2, \ldots, k_\ell)$-labellings of Γ is denoted by $\lambda_{k_1, k_2, \ldots, k_\ell}(\Gamma)$.

Distance labellings and their minimum spans have been investigated, for example, in [4, 6, 26, 27, 36]. The problem of finding $\lambda_{k_1, k_2, \ldots, k_\ell}(\Gamma)$ is motivated by the radio channel assignment problem and by the study of the scalability of optical networks (cf. [5, 37]). The problem is also important for the theory of classical graph colourings: for any graph Γ, it

is clear that the minimum span $\lambda_{1,\ldots,1}(\Gamma)$ is one less than the chromatic number of the ℓ-th power of Γ. This relation is valuable, since the problem of studying the chromatic numbers of powers of graphs is a significant problem in graph theory (see, for example, [2]). For illustrating examples of distance labellings and their minimum spans the reader can turn to Examples 4 and 5 in Section 2 of our paper.

The notion of a Cayley graph was introduced by Arthur Cayley to explain the structure of groups defined by a set of generators and relations. The literature on Cayley graphs of groups is vast and includes many deep results related to structural and applied graph theory; see for instance [3, 5, 29, 30, 33, 36].

Cayley graphs of semigroups, which we define next, are also very well known and have been studied since at least the 1960's [7, 35]. Let S be a semigroup, and let C be a nonempty subset of S. The *Cayley graph* $\mathrm{Cay}(S, C)$ of S with *connection set* C is defined as the graph with vertex set S and edge set $E = E(S, C)$ consisting of those ordered pairs (u, v) such that $cu = v$ for some $c \in C$. If the adjacency relation of $\mathrm{Cay}(S, C)$ is symmetric, then it becomes an undirected graph. Cayley graphs are the most important class of graphs associated with semigroups, as they have valuable applications (cf. [5, 18, 28]) and are related to automata theory (cf. [14, 16]). Many interesting results on Cayley graphs of various classes of semigroups have been obtained recently, for example, in [23, 25, 28, 31, 32, 34].

A semigroup is said to be *combinatorial* if all of its subgroups are singletons (see [1, 19] for recent results on this class of semigroups). Cayley graphs of combinatorial semigroups have not been considered in full generality; only small subclasses of this class have been investigated. A *band* is a semigroup entirely consisting of idempotents, i.e., elements x satisfying $x = x^2$. The class of combinatorial semigroups contains all bands and all Brandt semigroups over groups of order one. Bands play crucial roles in the structure theory of semigroups (see [11] and [16]). Cayley graphs of bands were explored in [8, 9, 12, 21, 22, 24], and Cayley graphs of Brandt semigroups were studied in [10, 20].

Our first main theorem shows that certain general restrictions on a certain formula expressing the span of a Cayley graph of a semigroup can force the semigroup to be combinatorial (Theorem 1) or to be a right zero band (Theorem 2), which are both well-known semigroup classes.

In addition to the results above, in Theorem 3 we describe all semigroups S and their subsets C such that the Cayley graph $\mathrm{Cay}(S, C)$ satisfies the restrictions that occur in our first two theorems. This yields the first description of the structure of all semigroups that possess Cayley graphs that are disjoint unions of complete graphs. An abstract characterisation of all Cayley graphs of semigroups which are disjoint unions of complete graphs was obtained in [15] in the language of equalities that hold for all elements of the semigroup. For several special classes of semigroups, a number of improvements and alternative element-wise characterisations have been obtained in [31] and [34]. However, the structure of such semigroups has not been characterised before, except in the special case of semigroups with bipartite Cayley graphs [15].

In Theorem 4, we give a characterisation of all graphs whose minimum spans can be determined by a formula with some of the restrictions that occur in the first three theorems.

Finally, we give examples to show that a straightforward upper bound for the minimum

spans of the underlying undirected graphs of Cayley graphs turns out to be sharp even for the class of combinatorial semigroups.

2   Main results

Our first main result shows that several restrictions on the minimum span of a Cayley graph are equivalent to the semigroup being combinatorial. As mentioned before, all our theorems will involve undirected Cayley graphs only. For any set or semigroup Q, we use the notation $|Q|$ for the cardinality of Q.

Theorem 1. *For any finite semigroup S, the following conditions are equivalent:*

(i) *S is combinatorial;*

(ii) *there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that, for each nonempty subsemigroup T of S and every nonempty subset C of T, if $\operatorname{Cay}(T, C)$ is undirected then*
$$\lambda_{k_1,\dots,k_\ell}(\operatorname{Cay}(T, C)) \leq F(k_1); \tag{1}$$

(iii) *there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that, for each nonempty subsemigroup T of S and every nonempty subset C of T, if $\operatorname{Cay}(T, C)$ is undirected, then*
$$\lambda_{k_1,\dots,k_\ell}(\operatorname{Cay}(T, C)) = F(k_1); \tag{2}$$

(iv) *there exists $\ell > 1$ such that, for each nonempty subsemigroup T of S and every nonempty subset C of T, if $\operatorname{Cay}(T, C)$ is undirected, then*
$$\lambda_{k_1,\dots,k_\ell}(\operatorname{Cay}(T, C)) = (|Cc| - 1)k_1 \tag{3}$$

for a fixed element c in C;

(v) *for each nonempty subsemigroup T of S and every nonempty subset C of T, if $\operatorname{Cay}(T, C)$ is undirected, then the equality*
$$\lambda_{k_1,\dots,k_\ell}(\operatorname{Cay}(T, C)) = (|Cc| - 1)k_1 \tag{4}$$

holds for every integer $\ell > 1$, all $k_1, \dots, k_\ell \in \mathbb{N}$, and every element c is C.

A band B is called a *right zero band*, if it satisfies the identity $xy = y$, for all $x, y \in B$. Our second result establishes several conditions on the minimum span that are equivalent to the semigroup being a right zero band.

Theorem 2. *For any finite semigroup S, the following conditions are equivalent:*

(i) *S is a right zero band;*

(ii) *there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that, for each nonempty subset C of S, $\mathrm{Cay}(S, C)$ is undirected and*

$$\lambda_{k_1,\dots,k_\ell}(\mathrm{Cay}(S, C)) \leq F(k_1); \tag{5}$$

(iii) *there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that, for each nonempty subset C of S, $\mathrm{Cay}(S, C)$ is undirected and*

$$\lambda_{k_1,\dots,k_\ell}(\mathrm{Cay}(S, C)) = F(k_1); \tag{6}$$

(iv) *there exists $\ell > 1$ such that, for each nonempty subset C of S, $\mathrm{Cay}(S, C)$ is undirected and*
$$\lambda_{k_1,\dots,k_\ell}(\mathrm{Cay}(S, C)) = (|Cc| - 1)k_1, \tag{7}$$

where c is a fixed element in C;

(v) *for each nonempty subset C of S, $\mathrm{Cay}(S, C)$ is undirected and the equality*

$$\lambda_{k_1,\dots,k_\ell}(\mathrm{Cay}(S, C)) = (|Cc| - 1)k_1 \tag{8}$$

holds for every integer $\ell > 1$, all $k_1, \dots, k_\ell \in \mathbb{N}$, and all elements c of C.

The next theorem gives a description of the structure of all semigroups whose Cayley graphs are disjoint unions of complete graphs, where it is assumed that the complete graphs contain all loops. At the same time it shows that this condition is equivalent to certain restrictions on the minimum span. If S is a semigroup with a subset C, then the subsemigroup generated by C in S is denoted by $\langle C \rangle$. Several conditions equivalent to the definition of a left group are given in Section 3 (see Lemma 1).

Theorem 3. *For any semigroup S and any nonempty subset C of S, the following conditions are equivalent:*

(i) *$CS = S$, $\langle C \rangle$ is a completely simple semigroup, and for each $c \in C$ the set Cc is a left group and a left ideal of $\langle C \rangle$;*

(ii) *there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that, for all $k_1, \dots, k_\ell \in \mathbb{N}$, $\mathrm{Cay}(S, C)$ is undirected and*

$$\lambda_{k_1,\dots,k_\ell}(\mathrm{Cay}(S, C)) \leq F(k_1); \tag{9}$$

(iii) *there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that, for all $k_1, \dots, k_\ell \in \mathbb{N}$, $\mathrm{Cay}(S, C)$ is undirected and*

$$\lambda_{k_1,\dots,k_\ell}(\mathrm{Cay}(S, C)) = F(k_1); \tag{10}$$

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(iv) *there exists $\ell > 1$ such that, for all $k_1, \ldots, k_\ell \in \mathbb{N}$, $\operatorname{Cay}(S, C)$ is undirected and*

$$\lambda_{k_1, \ldots, k_\ell}(\operatorname{Cay}(S, C)) = (|Cc| - 1)k_1, \tag{11}$$

where c is a fixed element in C;

(v) $\operatorname{Cay}(S, C)$ *is undirected, and for every integer $\ell > 1$ and all $k_1, \ldots, k_\ell \in \mathbb{N}$, the equality*

$$\lambda_{k_1, \ldots, k_\ell}(\operatorname{Cay}(S, C)) = (|Cc| - 1)k_1, \tag{12}$$

holds for all elements c of C;

(vi) $\operatorname{Cay}(S, C)$ *is a disjoint union of complete graphs.*

Example 1. The requirement that Cc be a left group in condition (i) of Theorem 3 cannot be omitted. For example, let $\mathbb{Z}_3 = \{e, g, g^2\}$ be the cyclic group of order 3, and let $C = \{e, g\}$. Then $\langle C \rangle = \mathbb{Z}_3$ is a completely simple semigroup and $C\mathbb{Z}_3 = \mathbb{Z}_3$, but $\operatorname{Cay}(\mathbb{Z}_3, C)$ is not a disjoint union of complete graphs.

Suppose that G is a group, I and Λ are nonempty sets, and $P = [p_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in G$ for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup* $M(G; I, \Lambda; P)$ over G with *sandwich-matrix* P consists of all triples $(h; i, \lambda)$, where $i \in I$, $\lambda \in \Lambda$, and $h \in G$, with multiplication defined by the rule

$$(h_1; i_1, \lambda_1)(h_2; i_2, \lambda_2) = (h_1 p_{\lambda_1 i_2} h_2; i_1, \lambda_2). \tag{13}$$

A semigroup is said to be *completely simple* if it has no proper ideals and has an idempotent minimal with respect to the natural partial order defined on the set of all idempotents by $e \le f \Leftrightarrow ef = fe = e$. It is well known that every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(G; I, \Lambda; P)$ over a group G (see [11], Theorem 3.3.1). Conversely, every semigroup $M(G; I, \Lambda; P)$ is completely simple.

A band B is called a *left zero band*, if it satisfies the identity $xy = x$, for all $x, y \in B$. Recall also that band B is called a *right zero band*, if it satisfies the identity $xy = y$, for all $x, y \in B$.

Example 2. The requirement that Cc be a left ideal of $\langle C \rangle$ in condition (i) of Theorem 3 cannot be omitted, even if it is still required to be a left group. For example, let $I = \{i_1, i_2\}$ be a left zero band, and let $G = \{e, g\}$ be the cyclic group of order 2. Then the direct product $G \times I$ is isomorphic to the completely simple semigroup $M(G; I, \Lambda; P)$, where $\Lambda = \{\lambda\}$, and $P = [e, e]$. Clearly, $G \times I$ is generated by the set $C = \{(g, i_1), (g, i_2)\}$. Take $c = (g, i_1)$. Then $Cc = \{(e, i_1), (e, i_2)\}$ is a left group, but $\operatorname{Cay}(G \times I, C)$ is not a disjoint union of complete graphs.

The next example shows that the complete graphs that occur in the disjoint union of condition (vi) of Theorem 3 may have different cardinalities. For $n \in \mathbb{N}$, the complete graph with n vertices is denoted by K_n.

Example 3. Let B be a finite left zero band, and let $B^0 = B \cup \{\theta\}$ be the semigroup obtained by adjoining zero θ to B. Then it is easily seen that

$$\mathrm{Cay}(B^0, B) = K_{|B|} \cup K_1 \tag{14}$$

is a disjoint union of two complete graphs, where $K_{|B|}$ and K_1 are the subgraphs induced in $\mathrm{Cay}(B^0, B)$ by the sets B and $\{\theta\}$, respectively.

The last main result of this paper describes all graphs whose minimum spans satisfy some of the restrictions that occur in the previous theorems.

Theorem 4. *For any finite undirected graph* $\Gamma = (V, E)$, *the following conditions are equivalent:*

(i) *there exist a function* $F : \mathbb{N} \to \mathbb{N}$ *and an integer* $\ell > 1$ *such that, for all* $k_1, \ldots, k_\ell \in \mathbb{N}$, *the minimum span of* Γ *satisfies the inequality*

$$\lambda_{k_1, \ldots, k_\ell}(\Gamma) \leq F(k_1); \tag{15}$$

(ii) *there exist a function* $F : \mathbb{N} \to \mathbb{N}$ *and an integer* $\ell > 1$ *such that, for all* $k_1, \ldots, k_\ell \in \mathbb{N}$, *the minimum span of* Γ *is determined by the formula*

$$\lambda_{k_1, \ldots, k_\ell}(\Gamma) = F(k_1); \tag{16}$$

(iii) Γ *is a disjoint union of complete graphs.*

We conclude this section with examples concerning the following straightforward upper bound on the minimum span of any undirected graph $\Gamma = (V, E)$:

$$\lambda_{k_1, \ldots, k_\ell}(\Gamma) \leq (|V| - 1) \max\{k_1, \ldots, k_\ell\}. \tag{17}$$

This inequality holds because the labelling that assigns $(i - 1) \max\{k_1, \ldots, k_\ell\} + 1$ to the i-th vertex (for any order of vertices) of Γ is an $L(k_1, \ldots, k_\ell)$-labelling with span $(|V| - 1) \max\{k_1, \ldots, k_\ell\}$.

The *underlying undirected graph* Γ^* of a directed graph $\Gamma = (V, E)$ is the graph with the same set V of vertices and with all undirected edges $\{u, v\}$ such that (u, v) or (v, u) is a directed edge of Γ. The following examples show that there exist combinatorial semigroups such that, for the underlying undirected graphs of their Cayley graphs, the upper bound (17) is sharp.

Example 4. Let B be a left zero band, and let C be a nonempty subset of B. Then

$$\lambda_{k_1, \ldots, k_\ell}(\mathrm{Cay}(B, C)^*) = k_1(|C| - 1) + \max\{k_1, k_2\} + k_2(|B| - |C| - 1). \tag{18}$$

Indeed, it follows from the definition of a left zero band that the set of edges of $\mathrm{Cay}(B, C)$ is the set of all edges (b, c), for all $b \in B$, $c \in C$. Therefore, for $x, y \in B$, $x \neq y$, we get

$$d(x, y) = \begin{cases} 1 & \text{if } x \in C \text{ or } y \in C \\ 2 & \text{if } x, y \in B \setminus C. \end{cases}$$

We can assign the labels $1, 1 + k_1, \ldots, 1 + k_1(|C| - 1)$ to the elements of C, and the labels $1 + k_1(|C| - 1) + \max\{k_1, k_2\}, 1 + k_1(|C| - 1) + \max\{k_1, k_2\} + k_2, \ldots, 1 + k_1(|C| - 1) + \max\{k_1, k_2\} + k_2(|B| - |C| - 1)$ to the elements of $B \backslash C$. This defines an $L(k_1, \ldots, k_\ell)$-labelling of $\mathrm{Cay}(B, C)$. The span of this labelling is equal to the right-hand side of (18), and it is clear that this is the minimum span. In the case where $k_1 = k_2$, equality (18) shows that the upper bound (17) is sharp.

The following example covers all combinatorial semigroups with zero. Notice that, for any combinatorial semigroup S, the semigroup $S^0 = S \cup \{\theta\}$ with zero θ adjoined in a standard fashion is combinatorial as well.

Example 5. Let S be a finite semigroup with zero θ, and let $C = \{\theta\}$. Then the set of edges of $\mathrm{Cay}(S, C)$ coincides with the set $\{(x, \theta) \mid x \in S\}$. Therefore

$$\lambda_{k_1, \ldots, k_\ell}(\mathrm{Cay}(S, C)^*) = k_1 + (|S| - 2)k_2. \tag{19}$$

Indeed, for $x, y \in B$, $x \neq y$, we have

$$d(x, y) = \begin{cases} 1 & \text{if } x = \theta \text{ or } y = \theta \\ 2 & \text{if } x, y \in S \setminus \{\theta\}. \end{cases}$$

Therefore we can assign the label 1 to θ and the labels $1 + k_1$, $1 + k_1 + k_2$, $1 + k_1 + 2k_2$, $\ldots$, $1 + k_1 + (|S| - 2)k_2$ to the nonzero elements of S. This defines an $L(k_1, \ldots, k_\ell)$-labelling of $\mathrm{Cay}(S, C)$ with the minimum possible span. This shows that $\lambda_{k_1, \ldots, k_\ell}(\mathrm{Cay}(S, C))$ is equal to the right-hand side of (19). In the case where $k_1 = k_2$, equality (19) shows that the bound (17) is sharp.

3 Preliminaries

We use standard terminology on graphs, Cayley graphs, groups and semigroups, and refer to [11, 16, 28] for more detailed explanations. The following notation and definitions are required for the proofs. A semigroup G is *periodic* if, for each $g \in G$, there exist positive integers m, n such that $g^m = g^{m+n}$. Obviously, every finite semigroup is periodic. A semigroup is said to be *right (left) simple* if it has no proper *right (left) ideals*. A semigroup is *left (right) cancellative* if $xy = xz$ (respectively, $yx = zx$) implies $y = z$, for all $x, y, z \in S$. A semigroup is called a *right (left) group* if it is right (left) simple and left (right) cancellative.

Lemma 1. ([17, Lemma 3.1]) *For any periodic semigroup S, the following are equivalent:*

(i) *S is right (left) simple;*

(ii) *S is a right (left) group;*

(iii) *S is isomorphic to the direct product of a right (left) zero band and a group;*

(iv) *S is a union of several of its left (right) ideals and each of these ideals is a group.*

A band B is called a *rectangular band* if it satisfies the identity $xyx = x$, for all $x, y \in B$.

Lemma 2. ([11, Theorem 1.1.3]) *Let B be a rectangular band. Then there exists a left zero band L and a right zero band R such that B is isomorphic to the direct product $L \times R$.*

It is very well known, and also follows immediately from Lemma 2, that every rectangular band B satisfies the identity $xyz = xz$, for all $x, y, z \in B$.

Let G be a group, $T = M(G; I, \Lambda; P)$, and let $i \in I$, $\lambda \in \Lambda$. Then we put

$$T_{*\lambda} = \{(h; j, \lambda) \mid h \in G, j \in I\},$$

$$T_{i*} = \{(h; i, \mu) \mid h \in G, \mu \in \Lambda\},$$

$$T_{i\lambda} = \{(h; i, \lambda) \mid h \in G\}.$$

Lemma 3. ([17, Lemma 3.2]) *Let G be a group, and let $T = M(G; I, \Lambda; P)$ be a completely simple semigroup. Then, for all $i, j \in I$, $\lambda, \mu \in \Lambda$, and $t = (h; i, \lambda) \in T$,*

(i) *the set $T_{*\lambda}$ is a minimal nonzero left ideal of T;*

(ii) *the set T_{i*} is a minimal nonzero right ideal of T;*

(iii) *$Tt = T_{*\mu}t = T_{*\lambda}$;*

(iv) *$tT = tT_{j*} = T_{i*}$;*

(v) *$t \in Tt \cap tT = T_{i\lambda}$;*

(vi) *the set $T_{i\lambda}$ is a left ideal of T_{i*} and a right ideal of $T_{*\lambda}$;*

(vii) *the set $T_{i\lambda}$ is a maximal subgroup of T isomorphic to G;*

(viii) *each maximal subgroup of T coincides with $T_{j\mu}$, for some $j \in I$, $\mu \in \Lambda$;*

(ix) *$M(G; I, \Lambda; P)$ is a right (left) group if and only if $|I| = 1$ (respectively, $|\Lambda| = 1$);*

(x) *if $T = M(G; I, \Lambda; P)$, then each $T_{*\lambda}$ is a left group, and each T_{i*} is a right group.*

It is well known that the Cayley graph $\mathrm{Cay}(G, C)$ of a group G is symmetric or undirected if and only if $C = C^{-1}$, that is $c \in C$ implies $c^{-1} \in C$. Undirected Cayley graphs of semigroups have been characterised in [13].

Proposition 1. ([13, Lemma 4]) *Let S be a semigroup, and let C be a subset of S, which generates a periodic subsemigroup $\langle C \rangle$. Then the following conditions are equivalent:*

(i) *$\mathrm{Cay}(S, C)$ is undirected;*

(ii) *$CS = S$, the semigroup $\langle C \rangle = M(H; I, \Lambda; P)$ is completely simple and, for each $(g; i, \lambda) \in C$ and every $j \in I$, there exists $\mu \in \Lambda$ such that $(p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu) \in C$.*

A graph Γ is said to be *connected* if its underlying undirected graph is connected. If, for each pair of vertices x, y of Γ, there exists a directed path from x to y, then Γ is said to be *strongly connected*. A (strong) connected component of Γ is a maximal strongly connected subgraph of Γ. Obviously, Γ is a vertex-disjoint union of its connected components.

Lemma 4. ([17, Lemma 5.1]) *Let S be a semigroup, C a subset of S, and $x \in S$. Let C_x be the set of all vertices y of $\mathrm{Cay}(S, C)$ such that there exists a directed path from x to y. Then C_x is equal to the right coset $\langle C \rangle x$.*

4 Proofs of the main results

Proof of Theorem 4. The implication (ii)$\Rightarrow$(i) is obvious. Let us prove implications (i)$\Rightarrow$(iii) and (iii)$\Rightarrow$(ii).

(i)$\Rightarrow$(iii): Suppose to the contrary that condition (i) holds, but (iii) is not satisfied. This means that there exist a function $F : \mathbb{N} \to \mathbb{N}$ and an integer $\ell > 1$ such that (15) holds for all $k_1, \ldots, k_\ell \in \mathbb{N}$, but the graph Γ is not a disjoint union of complete graphs. Clearly, any graph is a disjoint union of complete graphs if and only if $d(u, v) \in \{0, 1, \infty\}$ for every pair of vertices u, v of the graph. It easily follows that Γ contains a pair of vertices u, v at distance $d(u, v) = 2$ since the diameter of every connected component is not 1. Fix any value $k_1 \in \mathbb{N}$ and take $k_2 = F(k_1) + 1$. For any $L(k_1, \ldots, k_\ell)$-labelling f of Γ with span $\lambda_{k_1, \ldots, k_\ell}$, we get $|f(u) - f(v)| \geq k_2 > F(k_1)$. The maximality of $\lambda_{k_1, \ldots, k_\ell}$ implies that $|f(u) - f(v)| \leq \lambda_{k_1, \ldots, k_\ell}(\Gamma) \leq F(k_1)$. This contradiction shows that our assumption was wrong and (iii) must have been satisfied.

(iii)$\Rightarrow$(ii): Suppose that Γ is a disjoint union of complete graphs. Choose a connected component of Γ with the largest number of vertices, and denote the vertices of this component by $v_1, \ldots, v_n$, where $n \in \mathbb{N}$. Then the mapping f defined by $f(v_i) = k_1(i - 1) + 1$, for $i = 1, \ldots, n$, is an $L(k_1, \ldots, k_\ell)$-labelling of the connected component. Its span is equal to $|f(v_n) - f(v_1)| = |(k_1(n - 1) + 1) - 1| = k_1(n - 1)$. Similar labellings can be defined for all other connected components of Γ, since each of them has at most n vertices. This defines an $L(k_1, \ldots, k_\ell)$-labelling of Γ with span $k_1(n - 1)$. Taking the function $F(x) = x(n - 1)$, we see that equality (16) is always valid for F. This means that (ii) holds, which completes the proof. $\square$

Proof of Theorem 3. Implications (v)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii) are obvious. Implication (ii)$\Rightarrow$(vi) follows from Theorem 4. Let us prove implications (i)$\Rightarrow$(vi), (vi)$\Rightarrow$(i) and (vi)$\Rightarrow$(v).

(i)$\Rightarrow$(vi): Suppose that condition (i) holds. Then there exist a group G, sets I, Λ and $\Lambda \times I$ sandwich matrix P over G such that the subsemigroup $T = \langle C \rangle$ is isomorphic to the completely simple semigroup $M(G; I, \Lambda; P)$.

Take any element s in S and consider any pair of vertices $t_1 s, t_2 s$, where $t_1, t_2 \in T$. Given that $S = CS$, we can find $c \in C$ and $s_1 \in S$ such that $s = c s_1$. Since $c \in T$, there exist $g_c \in G$, $i_c \in I$ and $\lambda_c \in \Lambda$ such that $c = (g_c; i_c, \lambda_c)$. Likewise, $t_j = (g_j; i_j, \lambda_j)$ for $j = 1, 2$, $g_j \in G$, $i_j \in I$ and $\lambda_j \in \Lambda$. Since $S = CS$, we can also find $c_1 \in C$ and $s_2 \in S$ such that

$t_1 c = c_1 s_2$. Since the set Cc_1 is a left ideal of T, it follows that $Ct_1 c = Cc_1 s_2$ is also a left ideal of T. Lemma 3(ii) implies that $C_{i_2*} t_1 c = Ct_1 c \cap T_{i_2*}$. Hence $C_{i_2*} t_1 c$ is a left ideal of T_{i_2*}, because $Ct_1 c$ is a left ideal of T. In view of conditions (i) and (ii) of Lemma 3, $C_{i_2*} t_1 c$ is contained in $T_{i_2 \lambda_c}$. Hence $C_{i_2*} t_1 c$ is a left ideal of $T_{i_2 \lambda_c}$. However, $T_{i_2 \lambda_c}$ is a subgroup of T by Lemma 3(vii), and groups do not have proper left ideals. Therefore $C_{i_2*} t_1 c = T_{i_2 \lambda_c}$, and so $t_2 c = c_2 t_1 c$ for some $c_2 \in C_{i_2*}$. Thus, we get $t_2 s = t_2 c s_1 = c_2 t_1 c s_1 = c_2 t_1 s$. This means that $(t_1 s, t_2 s)$ is an edge of $\mathrm{Cay}(S, C)$. Since $t_1 s$ and $t_2 s$ were chosen arbitrarily, it follows that the set Ts induces a complete subgraph of $\mathrm{Cay}(S, C)$.

Clearly, S is a union of the sets Ts, for $s \in S$. Suppose that two sets Ts_1 and Ts_2 are not disjoint for some $s_1, s_2 \in S$. Then there exists $x \in Ts_1 \cap Ts_2$. Choose any $x_1 \in Ts_1$ and $x_2 \in Ts_2$. Since Ts_1 induces a complete subgraph in $\mathrm{Cay}(S, C)$, we see that (x, x_1) is an edge of $\mathrm{Cay}(S, C)$. Similarly, (x, x_2) is an edge of $\mathrm{Cay}(S, C)$. Hence $x_1, x_2 \in Cx \subseteq Tx$. However, Tx also induces a complete subgraph of $\mathrm{Cay}(S, C)$. It follows that (x_1, x_2) is an edge of $\mathrm{Cay}(S, C)$. Lemma 4 implies that $Tx = Ts_1 = Ts_2$.

Since the semigroup S is assumed to be finite, it follows that for some $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in S$ the graph $\mathrm{Cay}(S, C)$ is a disjoint union of the complete graphs induced by the sets $Ts_1, \ldots, Ts_n$. Thus, condition (vi) is satisfied.

(vi)$\Rightarrow$(i): Suppose that condition (vi) holds, that is, $\mathrm{Cay}(S, C)$ is a disjoint union of complete graphs. Then $\mathrm{Cay}(S, C)$ is undirected, and so Proposition 1 says that $CS = S$ and $\langle C \rangle$ is a completely simple semigroup. Take an arbitrary element c in C. It remains to verify that $CS = S$, and the set Cc is a left ideal of T and a left group.

Put $T = \langle C \rangle$. There exist a group G, sets I, Λ and $\Lambda \times I$ sandwich matrix P over G such that $T = M(G; I, \Lambda; P)$. Besides, $c = (g_c; i_c, \lambda_c)$ for some $g_c \in G$, $i_c \in I$ and $\lambda_c \in \Lambda$.

First, note that $\mathrm{Cay}(T, C)$ coincides with the subgraph of $\mathrm{Cay}(S, C)$ induced by the set T of vertices. Therefore it is also a disjoint union of complete graphs.

Clearly, Cc is equal to the set of vertices of the connected component of $\mathrm{Cay}(T, C)$ containing c. It follows from Lemma 4 that the set of vertices of the same connected component is equal to Tc. Since it is a complete graph with edges determined by the left multiplication of the elements of C, we see that it is also equal to the set $C^2 c$. Since $TC \subseteq T$, we get $TCc \subseteq Tc = Cc$. This means that the set Cc is a left ideal of T.

Condition (iii) of Lemma 3 implies that $Tc = T_{*\lambda_c}$. Condition (x) of Lemma 3 tells us that Tc is a left group, as required.

(vi)$\Rightarrow$(v): Suppose that condition (vi) holds. Since implication (vi)$\Rightarrow$(i) has already been proved, we can also use (i) and all properties that have been verified during the proof of (i). First of all note that, since $\mathrm{Cay}(S, C)$ is a union of complete graphs, it is undirected. Take any integer $\ell > 1$, any $k_1, \ldots, k_\ell \in \mathbb{N}$, and pick an arbitrary element $c \in C$. To prove the assertion, we have to verify that equality (12) is valid.

As it has been shown in the proof of the implication (i)$\Rightarrow$(vi) above, $\mathrm{Cay}(S, C)$ is a disjoint union of the graphs induced by the sets of vertices Ccs, for $c \in C$, $s \in S$. The number of vertices of each of these subgraphs does not exceed the cardinality $n = |Cc|$. As noted above in the discussion of equality (17), each complete graph K_n has an $L(k_1, \ldots, k_\ell)$-distance labelling with minimum span $(n-1)k_1$. It follows that (12) holds. This completes the proof. $\square$

The following easy lemma is used repeatedly in the proof of Theorem 1.

Lemma 5. *Let S be a semigroup with a subset C such that $\langle C \rangle$ is a band, $CS = S$, and let $x \in S$. Then there exists $e_x \in C$ such that $e_x x = x$.*

Proof. Take any element x in S. Since $CS = S$, it follows that there exist $e_x \in C$, $f_x \in S$ such that $e_x f_x = x$. Since S is a band, $e_x e_x = e_x$. Hence we get $e_x x = e_x e_x f_x = e_x f_x = x$, as required. $\square$

Proof of Theorem 1. Implications (v)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii) are obvious. Let us prove implications (ii)$\Rightarrow$(i) and (i)$\Rightarrow$(v).

(ii)$\Rightarrow$(i): Suppose to the contrary that condition (ii) holds, but S is not combinatorial. This means that S contains a subgroup G, which has an element g different from the identity e of G. Take $T = G$ and $C = \{g, g^{-1}\}$. The Cayley graph $\mathrm{Cay}(T, C)$ is undirected, because $C = C^{-1}$. Therefore inequality (1) holds, and so condition (ii) of Theorem 3 is satisfied. Hence $\mathrm{Cay}(T, C)$ is a disjoint union of complete graphs, where it is assumed that the complete graphs contain all loops. It follows that (e, e) is an edge of $\mathrm{Cay}(T, C)$. However, $ge \neq e$ and $g^{-1}e \neq e$. This contradiction shows that our assumption was wrong and S must have been combinatorial.

(i)$\Rightarrow$(v): Suppose that condition (i) holds. Choose a nonempty subsemigroup T of S and a nonempty subset C of T such that the Cayley graph $\mathrm{Cay}(T, C)$ is undirected. Take any integer $\ell > 1$, any $k_1, \ldots, k_\ell \in \mathbb{N}$, and fix an arbitrary element c is C. We have to verify that equality (4) is valid.

Proposition 1 implies that $CT = T$ and the semigroup $\langle C \rangle = M(H; I, \Lambda; P)$ is completely simple. Lemma 3(vii) tells us that for each $i \in I$, $\lambda \in \Lambda$, the set $T_{i\lambda}$ is isomorphic to G and is a maximal subgroup of $\langle C \rangle$. However, T is combinatorial, because it is a subsemigroup of S. Therefore $|G| = 1$. It follows that $\langle C \rangle$ is a rectangular band, and so it is isomorphic to the direct product of a left zero band I and a right zero band Λ, by Lemma 2.

Take an arbitrary element $x = (g; i, \lambda) \in \langle C \rangle$, where $i \in I$, $\lambda \in \Lambda$. We claim that $\langle C \rangle x = Cx$. To prove this, note that it easily follows from $\langle C \rangle = I \times \Lambda$ and the definitions of a right zero band and a left zero band, that

$$\langle C \rangle x = \{(j, \lambda) \mid j \in I\}. \tag{20}$$

For any $j \in I$ and $\mu \in \Lambda$, we get $(j, \mu)x \in \langle C \rangle_{j\lambda} \subseteq \langle C \rangle_{j*}$. If there existed $j \in I$ such that $C \cap \langle C \rangle_{j*} = \emptyset$, then it would follow that $\langle C \rangle \cap \langle C \rangle_{j*} = \emptyset$, which would be a contradiction. Therefore, for each $j \in I$, there exists $c_j \in C \cap \langle C \rangle_{j*}$. Hence we have $c_j x = (j, \lambda)$. It follows that

$$Cx = \{(j, \lambda) \mid j \in I\}. \tag{21}$$

Thus, (20) and (21) yield $\langle C \rangle x = Cx$.

Second, it also follows from equalities (20) and (21) that all cardinalities $|\langle C \rangle x| = |Cx|$ are equal to the same number $|I|$, and so they are all equal to $|Cc|$.

Third, (21) implies that the subgraph induced by the set Cx in $\mathrm{Cay}(T, C)$ is a complete graph.

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Next, we take an arbitrary element t in T, and claim that the subgraph induced by the set Ct in $\mathrm{Cay}(T,C)$ is a complete graph. Lemma 5 tells us that $e_t t = t$ for some $e_t \in C$. We have already shown in the preceding paragraph that Ce_t induces a complete subgraph in $\mathrm{Cay}(T,C)$. This means that, for any $c_1, c_2 \in C$, the pair $(c_1 e_t, c_2 e_t)$ is an edge of $\mathrm{Cay}(T,C)$, and so there exists $c_{c_1 c_2} \in C$ such that $c_{c_1 c_2} c_1 e_t = c_2 e_t$. We get $c_{c_1 c_2} c_1 t = c_{c_1 c_2} c_1 e_t t = c_2 e_t t = c_2 t$. This means that $(c_1 t, c_2 t)$ is an edge of the subgraph induced by Ct in $\mathrm{Cay}(T,C)$. Thus, this subgraph is indeed complete, with a loop attached to each of its vertices.

Further, suppose that for some elements t_1 and t_2 in T the sets Ct_1 and Ct_2 are not disjoint. Fix any element $z \in Ct_1 \cap Ct_2$, and consider an arbitrary pair of elements $z_1 \in Ct_1$ and $z_2 \in Ct_2$, where $z_1 = c_1 t_1$, $z_2 = c_2 t_2$, $c_1, c_2 \in C$. Since Ct_1 induces a complete subgraph in $\mathrm{Cay}(T,C)$, we can find $c' \in C$ such that $c' z_1 = c' c_1 t_1 = z$. Likewise, since Ct_2 induces a complete subgraph in $\mathrm{Cay}(T,C)$, there exists $c'' \in C$ such that $c'' z = c_2 t_2 = z_2$. Hence $c'' c' z_1 = z_2$. Lemma 5 implies that $e_{z_1} z_1 = z_1$ for some $e_{z_1} \in C$. As we have already proved in the preceding paragraph, the subgraph induced by Ce_{z_1} in $\mathrm{Cay}(T,C)$ is complete. Hence there exists $\bar{c} \in C$ such that $\bar{c} e_{z_1} = c'' c' z_1$. Therefore $\bar{c} z_1 = z_2$, which means that (z_1, z_2) is an edge of $\mathrm{Cay}(T,C)$. It follows that $z_1, z_2 \in Cz$. Thus $Ct_1 = Ct_2 = Cz$.

We have shown that any sets of the form Ct_1 and Ct_2, which are not disjoint, coincide. Since S is finite, it follows that there exist $n \in \mathbb{N}$ and elements $t_1, \ldots, t_n$ such that T is a disjoint union of the sets $Ct_1, \ldots, Ct_n$. We have also proved that each of the sets Ct_j, for $j = 1, \ldots, n$, induces a complete subgraph in $\mathrm{Cay}(T,C)$. Thus, $\mathrm{Cay}(T,C)$ is a disjoint union of subgraphs induced by the sets $Ct_1, \ldots, Ct_n$.

For each $j = 1, \ldots, n$, Lemma 5 allows us to find an element e_j such that $e_j t_j = t_j$. We get $Ct_j = Ce_j t_j$. Therefore the cardinality of the set Ct_j is equal to $|Ce_j \cdot t_j|$, which does not exceed $|Ce_j|$. As we have proved above, $|Ce_j| = |Cc|$. Thus, $\mathrm{Cay}(T,C)$ is a disjoint union of complete graphs each with at most $|Cc|$ vertices. As we have already noticed a few times during the proofs above, for every complete graph K_m with m vertices we have $\lambda_{k_1,\ldots,k_\ell}(K_m) = (m-1)k_1$. It follows that $\lambda_{k_1,\ldots,k_\ell}(\mathrm{Cay}(T,C)) = (|Cc|-1)k_1$. This completes the proof. $\square$

Proof of Theorem 2. Implications (v)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii) are obvious. Let us prove implications (i)$\Rightarrow$(v) and (ii)$\Rightarrow$(i).

(i)$\Rightarrow$(v): Suppose that S is a right zero band. Take an arbitrary nonempty subset C of S, any integer $\ell > 1$, and any positive integers $k_1, \ldots, k_\ell$. The set of edges of $\mathrm{Cay}(S,C)$ is equal to $E = \{(s,s) \mid s \in S\}$, because $cs = s$ for every $c \in C$, $s \in S$. It follows that $\mathrm{Cay}(S,C)$ is undirected and the function assigning 1 to all vertices of $\mathrm{Cay}(S,C)$ is an $L(k_1, \ldots, k_\ell)$-labelling. Therefore $\lambda_{k_1,\ldots,k_\ell}(\mathrm{Cay}(S,C)) = 0$. In the right zero band S, we have $Cc = \{c\}$, for all $c \in C$. Hence $(|Cc| - 1)k_1 = 0$, and so equality (8) holds for all $c \in C$. This means that condition (v) is satisfied, as required.

(ii)$\Rightarrow$(i): Suppose that condition (ii) of Theorem 2 holds. It shows immediately that, for any subset C of S, condition (ii) of Theorem 3 is satisfied. Since Theorem 3 has already been proved above, we can apply its condition (i) and conclude that for every subset C of S we have the following properties: $CS = S$, $\langle C \rangle$ is a completely simple semigroup, and for each $c \in C$ the set Cc is a left group and a left ideal of $\langle C \rangle$. Now let us apply these properties to

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several different subsets C of S.

First, consider the set $C = S$. We see that $S = \langle C \rangle = M(H; I, \Lambda; P)$ is a completely simple semigroup.

Second, pick any element $c \in S$ and consider the set $C = \{c\}$. There exist $g \in G$, $i \in I$, $\lambda \in \Lambda$ such that $c = (g; i, \lambda)$, and so $c \in S_{i*}$. As noted above, $CS = S$. However, condition (ii) of Lemma 3 tells us that S_{i*} is a left ideal of S; whence $cS \subseteq S_{i*}$. It follows that $S = S_{i*}$, which means that $I = \{i\}$. Condition (x) of Lemma 3 tells us that $S = S_{i*}$ is a right group. By Lemma 1(iii), S is isomorphic to the direct product of a right zero band and a group. To simplify notation, we may assume that Λ is a right zero band and $S = G \times \Lambda$.

Third, choose any element $c = (g, \mu) \in G \times \Lambda = S$, where $g \in G$, $\mu \in \Lambda$, and consider the set $C = \{(g, \mu)\}$. Evidently, $Cc = \{(g^2, \mu)\}$. As noted above, we know that Cc is a left ideal of $\langle C \rangle$. Hence $c(g^2, \mu) = (g^3, \mu) \in Cc$, and so $g^2 = g^3$. It follows that g is equal to the identity e of G. Since g was chosen as an arbitrary element of G, we see that $G = \{e\}$. Therefore $S \cong \Lambda$ is a right zero band. Thus, condition (i) of Theorem 2 holds. This completes the proof. $\square$

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